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GRAVITATIONALLY DRESSED CONFORMAL FIELD THEORY AND EMERGENCE OF LOGARITHMIC OPERATORS

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ABSTRACT

We study correlation functions in two-dimensional conformal field theory coupled to induced gravity in the light-cone gauge. Focussing on the fermion four-point function, we display an unexpected non-perturbative singularity structure: coupling to gravity *qualitatively* changes the perturbative $(x_1 - x_2)^{-1}(x_3 - x_4)^{-1}$ singularity into a logarithmic one plus a non-singular piece. We argue that this is related to the appearance of new logarithmic operators in the gravitationally dressed operator product expansions.

We also show some evidence that non-conformal but integrable models may remain integrable when coupled to gravity.

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In this letter we consider the gravitational dressing in a two-dimensional conformal field theory coupled to two-dimensional quantum gravity. This problem is important not only for conformal models on random surfaces and non-critical strings but also as a starting point for understanding the effect of gravitational dressing in renormalizable but not conformally invariant models. One of the interesting possibilities is the appearance of new states in the theory due to the inclusion of gravity. This phenomenon is non-perturbative and it is the purpose of this letter to explore it to some extent and to show that the new states indeed appear. To find these states and examine their unusual properties we shall study the gravitationally dressed 4-point function using the light-cone formulation of two-dimensional quantum gravity [1].

To begin with, following [2-3], we derive a differential equation for the gravitationally dressed fermion n -point function. In light-cone gauge, gravity is taken into account by adding $\int d^2x h_{++}(x)T_{--}(x)$ to the action so that the gravitationally dressed correlation functions are

$$\langle\langle\phi_1(x_1)\dots\phi_n(x_n)\rangle\rangle = \int \mathcal{D}h_{++} \langle\phi_1(x_1)\dots\phi_n(x_n)\rangle \exp\left(i \int d^2x h_{++}(x)T_{--}(x)\right) . \quad (1)$$

If ϕ_j has conformal dimension Δ_j , i.e. if under a reparametrization

$$\delta_\epsilon \phi_j = \epsilon_+ \partial_- \phi_j + \Delta_j (\partial_- \epsilon_+) \phi_j \quad (2)$$

then the corresponding integrated Ward identity is [2]

$$\begin{aligned} & \gamma \langle\langle h_{++}(z) \phi_1(x_1) \dots \phi_n(x_n) \rangle\rangle + \sum_{j=1}^n \left[\frac{(z^- - x_j^-)^2}{z^+ - x_j^+} \frac{\partial}{\partial x_j^-} - 2\Delta_j \frac{z^- - x_j^-}{z^+ - x_j^+} \right] \langle\langle \phi_1(x_1) \dots \phi_n(x_n) \rangle\rangle \\ & = 0 . \end{aligned} \quad (3)$$

Here Δ_j are the conformal dimensions in the presence of gravity as given by the gravitationally dressed two-point functions, and γ is related to the level k of the gravitational $Sl(2, \mathbf{R})$ current algebra [1] by

$$\gamma = k + 2 = \frac{1}{12} \left(c - 13 - \sqrt{(c-1)(c-25)} \right) \quad (4)$$

where c is the total central charge of the matter coupling to gravity.

Using now the quantum equations of motion (i.e. Schwinger-Dyson equations) for a left-handed fermion in the presence of gravity

$$\partial_+ \psi =: h_{++} \partial_- \psi : + \Delta : (\partial_- h_{++}) \psi : \quad (5)$$

one obtains for the n fermion correlation function

$$\left\{ \gamma \frac{\partial}{\partial z^+} + \sum_{j=2}^n \left[\frac{(z^- - x_j^-)^2}{z^+ - x_j^+} \frac{\partial}{\partial z^-} \frac{\partial}{\partial x_j^-} - 2\Delta \frac{z^- - x_j^-}{z^+ - x_j^+} \left(\frac{\partial}{\partial z^-} - \frac{\partial}{\partial x_j^-} \right) - \frac{2\Delta^2}{z^+ - x_j^+} \right] \right\} \langle \langle \psi(z) \psi(x_2) \dots \psi(x_n) \rangle \rangle = 0 . \quad (6)$$

One may check that this gives the two-point function correctly as

$$\langle \langle \psi(z) \psi(x) \rangle \rangle = \frac{1}{(z^- - x^-)^{2\Delta} (z^+ - x^+)^{2\Delta-1}} \quad (7)$$

with

$$\Delta - \frac{1}{2} = -\frac{\Delta(1-\Delta)}{k+2} . \quad (8)$$

We now concentrate on the fermion four-point function. Conformal invariance implies

$$\begin{aligned} G_4(w, x, y, z) &\equiv \langle \langle \psi_i(w) \psi_i(x) \psi_j(y) \psi_j(z) \rangle \rangle \\ &= \frac{f(t^-, t^+)}{(w^- - x^-)^{2\Delta} (w^+ - x^+)^{2\Delta-1} (y^- - z^-)^{2\Delta} (y^+ - z^+)^{2\Delta-1}} . \end{aligned} \quad (9)$$

We have added colour indices for the fermions, and if $i \neq j$, then obviously $f = 1 + \mathcal{O}(1/\gamma)$. The anharmonic ratio t is given as usual by

$$t \equiv t^- = \frac{(w^- - y^-)(x^- - z^-)}{(w^- - z^-)(x^- - y^-)} \quad (10)$$

and similarly for $\bar{t} \equiv t^+$. Inserting the ansatz (9) into eq. (6) with $n = 4$ leads after some algebra to

$$\left[\gamma \bar{t} \partial_{\bar{t}} + \frac{1-t}{1-\bar{t}} (\bar{t} - t) \partial_{\bar{t}} t \partial_t + (1-4\Delta) t \partial_t + 2\Delta^2 \frac{t+1}{t-1} \right] f(t, \bar{t}) = 0 . \quad (11)$$

Note that for $\Delta = 1$ this reproduces the equation derived in [3] for the four-current correlation function.

This partial non-linear differential equation (11) seems too complicated to be solved in full generality. In principle, it can be solved in a perturbative series order by order in $1/\gamma \sim 1/c$. To fix the integration ambiguity it is useful to compare with a direct Feynman diagram computation for the four-point function. The latter gives (for colour indices $i \neq j$)

$$f = 1 - \frac{1}{2\gamma} \frac{t+1}{t-1} \log t\bar{t} + \mathcal{O}(1/\gamma^2) = 1 - \frac{2\Delta^2}{\gamma} \frac{t+1}{t-1} \log t\bar{t} + \mathcal{O}(1/\gamma^2) . \quad (12)$$

Although not obvious on the form (12) the momentum space four-point function (with the external legs removed) vanishes at this order when the fermions are on shell. Indeed, it is easy to see that it is proportional to

$$\frac{1}{\gamma} (p+p')_-(q+q')_- - \frac{(p-p')_+}{(p-p')_-^3}$$

where p , p' and q , q' are the initial and final momenta of the first and second fermions. For left fermions one has the on-shell condition $p_+ = p'_+ = 0$ (and the same for q) and one gets no contribution to the S -matrix at this order. We have verified that this remains true at the next order, including two graviton exchanges. In other words, up to this order, the S -matrix for gravitational scattering of two left fermions is unity. If this turns out to be true at all orders one would have an interesting situation: Consider the Gross-Neveu model coupled to gravity. In the light-cone gauge only the left fermions couple to gravity. Left-left scattering would then be elastic, as is right-right scattering (anyhow) and left-right scattering (by kinematics). Thus any two-particle scattering would be elastic. This is an encouraging result to speculate that the integrability of the Gross-Neveu model might survive coupling to gravity. One can even try to formulate a gravitationally dressed Bethe-Ansatz which may help to solve exactly integrable models coupled to induced gravity. Details will be given elsewhere [4].

The main difficulty in solving the differential equation (11) is the factor $1/(1-\bar{t})$. If one considers the vicinity of $t = 1$, this difficulty disappears, and (11) becomes

$$\left[\gamma \bar{t} \partial_{\bar{t}} + (t-1) \partial_t t \partial_t + (1-4\Delta) \partial_t + \frac{4\Delta^2}{t-1} \right] f_1(t, \bar{t}) = 0 \quad (13)$$

where the subscript 1 on f is to remind us that $f \sim f_1$ only in the vicinity of $t = 1$. Equation

(13) can be solved exactly. First, perturbation in $1/\gamma$ leads to (matching with (12))

$$f_1(t, \bar{t}) = \sum_{n=0}^{\infty} \frac{[(2\Delta)_n]^2}{n!} g^n \quad , \quad g = -\frac{\log \bar{t}}{\gamma(t-1)} \quad (14)$$

where $a_n \equiv a(a+1)(a+2)\dots(a+n-1)$. At each order in $1/\gamma$ one can of course replace $\log \bar{t}$ by $\log t\bar{t}$ in g , as suggested by (12).[★] The series (14) has zero radius of convergence, but its Borel transform can be recognized as the hypergeometric function

$$B[f_1](z) = \sum_{n=0}^{\infty} \frac{[(2\Delta)_n]^2}{n!n!} z^n = F(2\Delta, 2\Delta, 1; z) . \quad (15)$$

Inverting the Borel transform gives the resummed function $f_1(g)$ in terms of the Whittaker function. Alternatively, one can directly observe that (14) coincides up to an overall factor with the asymptotic expansion of the Whittaker function. Hence [5]

$$f_1(g) = (-1/g)^{2\Delta} \Psi(2\Delta, 1; -1/g) = (-1/g)^{2\Delta-1/2} e^{-1/2g} W_{1/2-2\Delta, 0}(-1/g) . \quad (16)$$

Here W is the Whittaker function and Ψ is a solution to the degenerate hypergeometric equation [5]. Indeed, although the equation (13) has many solutions, if one uses an ansatz with f_1 only depending on t and \bar{t} through g , then equation (13) becomes

$$\left\{ g^2 \frac{d^2}{dg^2} + [(1+4\Delta)g-1] \frac{d}{dg} + 4\Delta^2 \right\} f_1(g) = 0 . \quad (17)$$

Setting $f_1(g) = (-1/g)^{2\Delta} u(-1/g)$ one sees that $u(x)$ satisfies the degenerate hypergeometric equation

$$xu''(x) + (b-x)u'(x) - au(x) = 0 \quad (18)$$

with $b = 1$ and $a = 2\Delta$. Perturbation theory has told us which of the two independent solutions to choose, namely $u(x) = \Psi(2\Delta, 1; x)$.

★ Then f_1 is no longer an exact solution of (13) but an exact solution of another equation, differing from (13) only by higher order terms in $(t-1)$, just as (13) differs from (11) by higher order terms in $(t-1)$ anyhow.

Having the perfectly non-perturbative expression (16) for $f_1(g)$, we can now investigate its behaviour for large g , which is just the series expansion of $\Psi(2\Delta, 1; x)$ for small x [5]:

$$f_1(g) = \left(-\frac{1}{g}\right)^{2\Delta} \sum_{k=0}^{\infty} \frac{\Gamma(2\Delta + k)}{[k!\Gamma(2\Delta)]^2} \left[2\psi(k+1) - \psi(2\Delta + k) - \log\left(-\frac{1}{g}\right) \right] \left(-\frac{1}{g}\right)^k \quad (19)$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$. Recall that $g = -\log \bar{t}/[\gamma(t-1)]$, hence large g means $t \rightarrow 1$ (for fixed \bar{t}), so this is the limit where $f \sim f_1$. Although (19) is the exact asymptotic for f_1 , it gives only the leading order for f as $t \rightarrow 1$:

$$f(t, \bar{t}) \sim \left(\frac{\gamma(t-1)}{\log \bar{t}}\right) \frac{1}{\Gamma(2\Delta)} \left[\psi(1) - \log\left(\frac{\gamma(t-1)}{\log \bar{t}}\right) + \mathcal{O}((t-1), (t-1)\log(t-1)) \right]. \quad (20)$$

What does this mean for the fermion four-point function (9)? Since $t-1 = \frac{(w^- - x^-)(y^- - z^-)}{(w^- - z^-)(x^- - y^-)}$, one has $t \rightarrow 1$ if either $w^- \rightarrow x^-$ or $y^- \rightarrow z^-$, i.e. when two fermion operators of the same colour approach each other. Inserting (20) into (9) then gives

$$\begin{aligned} G_4(w, x, y, z) &\sim \frac{\gamma^{2\Delta}}{\Gamma(2\Delta)} [(w^- - z^-)(x^- - y^-)]^{-2\Delta} [\log \bar{t}]^{-2\Delta} [(w^+ - x^+)(y^+ - z^+)]^{1-2\Delta} \\ &\times \left[\psi(1) + \log\left(\frac{\log \bar{t}}{\gamma}\right) - \log \frac{(w^- - x^-)(y^- - z^-)}{(w^- - z^-)(x^- - y^-)} \right]. \end{aligned} \quad (21)$$

It is important to realize that we work in Minkowski space so that we can take $t \rightarrow 1$, keeping $\bar{t} \neq 1$ fixed. Rather surprisingly, the four-point function (21) no longer contains the perturbative singularity $\sim (w^- - x^-)^{-2\Delta}(y^- - z^-)^{-2\Delta}$, but resumming the series has transformed it into a logarithmic singularity, plus a non-singular part !

Mathematically, the origin of the logarithm can be traced to the degenerate hypergeometric equation (18) satisfied by $u(x) = x^{-2\Delta} f_1(-1/x)$. For generic parameter b it has two independent solutions [5] $\Phi(a, b; x)$ and $x^{1-b}\Phi(a+1-b, 2-b; x)$. Obviously, for $b \rightarrow 1$ the second solution generates $\log x \Phi(a, b; x)$, among others. This is a well-known phenomenon in the theory of ordinary linear differential equations.

Physically however, it was quite unexpected that turning on gravity ($1/\gamma \neq 0$), even infinitesimally weakly, completely changes the singularity structure: this is a truly non-perturbative phenomenon.

The appearance of logarithms in correlation functions had been noticed before in [6] where the WZW model based on the supergroup $GL(1,1)$ was discussed. Later these logarithms were discussed in the $c = -2$ model and other nonunitary non-minimal models in [7]. There it has been argued that the emergence of the logarithms in correlation functions are due to the existence of new operators in the operator product expansion with a new, “logarithmic” behaviour. The anomalous dimensions of these new operators are degenerate with those of the usual primary operators. Then one no longer can completely diagonalize the Virasoro operator L_0 and the new operators together with the standard ones are the basis of the Jordan cell for L_0 . More precisely, in the case of two operators \tilde{O}_n and O_n with degenerate anomalous dimensions Δ_n the operator product expansion now takes the form

$$\phi(x)\phi(0) = \sum_n x^{\Delta_n - 2\Delta_\phi} \left[\tilde{O}_n + \dots + \log(x)O_n + \dots \right]. \quad (22)$$

These operators also have unusual OPEs with the stress-energy tensor:

$$T(z)\tilde{O}_n(0) = \frac{\Delta_n}{z^2}\tilde{O}_n(0) + \frac{1}{z^2}O_n(0) + \frac{1}{z}\partial_z\tilde{O}_n(0) \quad (23)$$

in particular

$$L_0|O_n\rangle = \Delta_n|O_n\rangle, \quad L_0|\tilde{O}_n\rangle = \Delta_n|\tilde{O}_n\rangle + |O_n\rangle \quad (24)$$

This allows us to have logarithmic terms in correlation functions without spoiling the conformal invariance. Using these rules one can extract information about the OPEs in the gravitaionally dressed theory from the correlation function (21). Obviously, as far as the left dimensions are concerned, eq. (21) corresponds to $\Delta_n - 2\Delta_\psi = 0$. More details will be given elsewhere.

Let us note that in the $c = 1$ string one also encounters an analogous phenomenon: As shown in [8], in the linear approximation, the tachyon field \mathcal{T} obeys the following equation

$$-\partial_t^2\mathcal{T} + \partial_\phi^2\mathcal{T} + 2\sqrt{2}\partial_\phi\mathcal{T} + 2\mathcal{T} + .. = 0 \quad (25)$$

where ϕ is the Liouville field. For solutions independent of t the characteristic equation $\lambda^2 + 2\sqrt{2}\lambda + 2 = (\lambda + \sqrt{2})^2 = 0$ has degenerate roots, and the two solutions are $\exp(-\sqrt{2}\phi)$ and $\phi \exp(-\sqrt{2}\phi)$. Considering correlation functions of the type $\langle \mathcal{T}(z_1)\mathcal{T}(z_2)\dots \rangle$ one gets

structures like z^α and $\log z z^\alpha$, revealing again in the conformal gauge signs of the logarithmic terms we found in the light-cone formulation of 2-dimensional gravity. In both cases one has degeneracy of the anomalous dimensions of some conformal fields.

In conclusion, we have derived the exact differential equation (11) for the gravitationally dressed fermion four-point function. In the vicinity of $t \rightarrow 1$ this equation simplifies, and the simplified equation (13) could be solved to all orders in $1/\gamma \sim 1/c$. We could resum the divergent perturbation series, and thus study the singularity structure of the four-point function for $t \rightarrow 1$ (non-perturbative regime). Rather surprisingly, non-perturbative effects of gravity have changed the $(w^- - x^-)^{-1}(y^- - z^-)^{-1}$ singularity into a logarithmic one plus a regular term, thus indicating the appearance of new logarithmic operators.

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